

AN ASCENDING HNN EXTENSION OF A FREE GROUP INSIDE $\mathrm{SL}_2\mathbb{C}$

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(Communicated by Ronald A. Fintushel)

ABSTRACT. We give an example of a subgroup of $\mathrm{SL}_2\mathbb{C}$ which is a strictly ascending HNN extension of a non-abelian finitely generated free group F . In particular, we exhibit a free group F in $\mathrm{SL}_2\mathbb{C}$ of rank 6 which is conjugate to a proper subgroup of itself. This answers positively a question of Drutu and Sapir (2005). The main ingredient in our construction is a specific finite volume (non-compact) hyperbolic 3-manifold M which is a surface bundle over the circle. In particular, most of F comes from the fundamental group of a surface fiber. A key feature of M is that there is an element of $\pi_1(M)$ in $\mathrm{SL}_2\mathbb{C}$ with an eigenvalue which is the square root of a rational integer. We also use the Bass-Serre tree of a field with a discrete valuation to show that the group F we construct is actually free.

1. INTRODUCTION

Suppose $\phi: F \rightarrow F$ is an injective homomorphism from a group F to itself. The associated HNN extension

$$H = \langle G, t \mid tgt^{-1} = \phi(g) \text{ for } g \in G \rangle$$

is said to be *ascending*; this extension is called *strictly ascending* if ϕ is not onto. In [DS], Drutu and Sapir give examples of residually finite 1-relator groups which are not linear; that is, they do not embed in $\mathrm{GL}_n K$ for any field K and dimension n . All of their examples are ascending HNN extensions of free groups, and indeed this is how they know the groups are residually finite (by [BS]). Motivated by looking at the $\mathrm{SL}_2\mathbb{C}$ representations of their examples, they asked whether $\mathrm{SL}_2\mathbb{C}$ contains a strictly ascending HNN extension of a non-abelian free group [DS, Problem 8]. In particular, they asked if there is a non-abelian free group F in $\mathrm{SL}_2\mathbb{C}$ which is conjugate to a proper subgroup of itself. In this note, we answer their question in the affirmative.

1.1. Theorem. *The group $\mathrm{SL}_2\mathbb{C}$ contains a strictly ascending HNN extension of a free group of rank 6.*

We first outline a general method of finding such groups, and then describe a specific example in more detail.

Received by the editors February 18, 2005 and, in revised form, June 7, 2005.

2000 *Mathematics Subject Classification.* Primary 20E06; Secondary 57Mxx.

Key words and phrases. Ascending HNN extension, $\mathrm{SL}_2\mathbb{C}$, hyperbolic 3-manifold.

Both authors were partially supported by the U.S. N.S.F. and the Sloan Foundation.

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1.2. Surface bundles over S^1 . A large part of our example comes from the fundamental group of a hyperbolic 3-manifold which fibers over the circle. Let Σ be an open surface of finite type, that is, a closed surface with a non-empty finite set removed. Assume Σ has negative Euler characteristic. Given a homeomorphism $\phi : \Sigma \rightarrow \Sigma$ we can form the mapping torus

$$M_\phi = \Sigma \times [0, 1] / (s, 1) \sim (\phi(s), 0).$$

Thurston showed that provided ϕ is pseudo-Anosov (i.e. ϕ is not homotopic to a finite order or reducible homeomorphism), M_ϕ admits a *complete* hyperbolic structure [Thu, Otal]. By Mostow rigidity, this hyperbolic structure is in fact unique.

If M_ϕ is orientable, then after choosing an orientation, the hyperbolic structure determines a discrete, faithful representation

$$\rho : \pi_1(M_\phi) \rightarrow \text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2\mathbb{C}.$$

It turns out that ρ always lifts to a representation into $\text{SL}_2\mathbb{C}$ (see e.g. [Cul]), and henceforth we will regard $\text{SL}_2\mathbb{C}$ as the target of ρ .

1.3. The construction. Now $\pi_1(M_\phi)$ is an HNN extension

$$\pi_1(M_\phi) = \langle \pi_1(\Sigma), t \mid t\alpha t^{-1} = \phi_*(\alpha) \text{ for all } \alpha \in \pi_1(\Sigma) \rangle.$$

Since Σ is an open surface of finite type, its fundamental group $\pi_1(\Sigma)$ is a finitely generated free group. Take $F = \rho(\pi_1(\Sigma))$ which is a free subgroup of $\text{SL}_2\mathbb{C}$. Note that for any $\beta \in \pi_1(M_\phi)$, the element $\rho(\beta)$ conjugates F to itself.

We now restrict our attention to M_ϕ with a very special property; we will give an example of such an M_ϕ in Section 3.

1.4. Requirement. There is a $\beta \in \pi_1(M_\phi) \setminus \pi_1(\Sigma)$ such that $\rho(\beta)$ has trace $\pm(\sqrt{n} + 1/\sqrt{n})$ for some integer $n \geq 2$.

Equivalently, we want $\rho(\beta)$ to have eigenvalues $\pm\sqrt{n}$ and $\pm 1/\sqrt{n}$. The image of $\rho(\beta)$ in $\text{SL}_2\mathbb{C}$ is a hyperbolic isometry of \mathbb{H}^3 , and has two fixed points in $\partial\mathbb{H}^3 = \mathbb{CP}^1 = \mathbb{C} \cup \infty$. After conjugating ρ , we can assume these fixed points are 0 and ∞ , and $\rho(\beta)$ acts on \mathbb{CP}^1 by $z \mapsto nz$. For $t \in \mathbb{C}$, let μ_t denote the parabolic element of $\text{SL}_2\mathbb{C}$ which acts on \mathbb{CP}^1 by $z \mapsto z + t$. Note that $\rho(\beta)$ conjugates μ_t to $(\mu_t)^n$.

For each t , define $G_t = \langle F, \mu_t \rangle$. Then G_t is finitely generated, and $\rho(\beta)$ conjugates it to a proper subgroup of itself. The following lemma, proved in the next section, shows that G_t is free for generic values of t .

1.5. Lemma. *Let F be a finitely generated subgroup of $\text{SL}_2\mathbb{C}$. Consider the group $G_t = \langle F, \mu_t \rangle$ for some $t \in \mathbb{C}$. Suppose that no non-trivial element of F fixes $\infty \in \mathbb{CP}^1$. Then for generic t , the group G_t is isomorphic to the free product $F * \mathbb{Z}$.*

For the action of the fundamental group of a finite volume hyperbolic manifold on \mathbb{CP}^1 , the stabilizer of any point is either empty, an infinite cyclic group consisting of hyperbolic elements, or \mathbb{Z}^2 consisting of parabolic elements. Now $\rho(\beta)$ is not parabolic, as its trace is not ± 2 . Thus the stabilizer of ∞ in $\pi_1(M_\phi)$ is infinite cyclic. So if some $f \in F$ also fixed ∞ , then $f^n = \rho(\beta)^m$ for suitable non-trivial powers n and m . Since no power of β lies in $\pi_1(\Sigma)$, the element $\rho(\beta)$ has no fixed point in common with any element of F . Thus the lemma applies in our context and G_t is free for most choices of t .

Thus, given a hyperbolic 3-manifold satisfying the requirement, we get a free group G_t which is conjugate to a proper subgroup of itself by $\rho(\beta)$. Then the subgroup $H = \langle G_t, \rho(\beta) \rangle$ of $\mathrm{SL}_2\mathbb{C}$ is isomorphic to the ascending HNN extension of G_t by the map induced via conjugation by $\rho(\beta)$.

This completes our construction and proves Theorem 1.1 modulo two things: proving the lemma, and exhibiting an M_ϕ satisfying Requirement 1.4 where $\pi_1(\Sigma)$ has rank 5. These remaining tasks are carried out in the next two sections. The reader may wonder how plausible it is to expect a manifold satisfying Requirement 1.4. For any finite volume hyperbolic 3-manifold, the geometric representation $\rho: \pi_1(M) \rightarrow \mathrm{SL}_2\mathbb{C}$ can be conjugated so that its image lies in SL_2L for some finite extension L of \mathbb{Q} . This is because if one looks at representations $\pi_1(M_\phi) \rightarrow \mathrm{SL}_2\mathbb{C}$ where the cusp group acts by parabolics, then the representation ρ coming from the hyperbolic structure is an isolated point if we mod out by conjugation [GR]. The set of conjugacy classes of such representations is an algebraic variety defined over \mathbb{Q} , and hence isolated points have coordinates in a number field. Thus it should not be surprising that the trace of β has a particular value in a quadratic field.

1.6. *Remark.* For our examples, the monomorphism of G_t is necessarily reducible, i.e. it preserves a free product decomposition of G_t into proper factors. Mark Sapir asked whether there are also examples where this monomorphism is irreducible.

1.7. *Remark.* For our examples G_t is necessarily indiscrete. In fact this must always be the case. Potyagailo and Ohshika proved that any topologically tame non-elementary 3-dimensional Kleinian group can not be conjugate in $\mathrm{SL}_2\mathbb{C}$ to a proper subgroup [OP]. Since all finitely generated Kleinian groups are topologically tame [Agol, CG], this implies that there are no examples where the free group base of the HNN extension is discrete.

2. PROOF OF THE LEMMA

Proof of Lemma 1.5. The basic idea here is to use the Bass-Serre tree to show that the “universal” group $\langle F, \mu_t \rangle$, where t is viewed as a parameter, is isomorphic to the free product $F * \mathbb{Z}$. It will then easily follow that G_t is also isomorphic to $F * \mathbb{Z}$ for generic choices of t .

For notational convenience, we change our parameterization of the parabolic μ_t from $z \mapsto z + t$ to $z \mapsto z + 1/t$. In matrix form

$$\mu_t = \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}.$$

Now let K be the field of rational functions over \mathbb{C} . Consider the homomorphism $i: F * \mathbb{Z} \rightarrow \mathrm{SL}_2K$ which is just the inclusion of $\mathrm{SL}_2\mathbb{C}$ into SL_2K on the first factor, and sends the generator of \mathbb{Z} to the above parabolic matrix. We claim that i is injective.

Consider the discrete valuation on K whose value on a rational function $f(t)$ is the order of vanishing of $f(t)$ at $t = 0$. Let \mathcal{O} be the valuation ring of K , i.e. the set of rational functions which do not have a pole at 0. Note that \mathcal{O} is a local ring, and $t\mathcal{O}$ is its maximal ideal. Then the action of SL_2K on the Bass-Serre tree associated to this valuation gives us a splitting of SL_2K as a free product with amalgamation as follows (see [Ser, §II.1.4] or [Sha, §3] for details). Let $A = \mathrm{SL}_2\mathcal{O}$

and $B = XAX^{-1}$ be the conjugate of A by

$$X = \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 K, \text{ i.e. } B = \begin{pmatrix} a & t^{-1}b \\ tc & d \end{pmatrix} \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mathcal{O}.$$

If we set

$$C = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \mathcal{O} \mid c \in t\mathcal{O} \right\},$$

then $\mathrm{SL}_2 K = A *_C B$. (See the top of page 81 of [Ser, §II.1.4] for this precise splitting in the analogous case of $\mathrm{SL}_2 \mathbb{Z}[1/p]$.)

Note that the map $i: F * \mathbb{Z} \rightarrow \mathrm{SL}_2 K$ takes F into the first factor of the amalgam and takes \mathbb{Z} into the other. The condition that no element of F fix ∞ exactly corresponds to the condition that the image $i(F)$ in $\mathrm{SL}_2 K$ does not intersect the edge group C joining the two factors of the amalgam. As $i(\mathbb{Z})$ is also disjoint from C , the subgroup of $\mathrm{SL}_2 K$ generated by $i(F)$ and $i(\mathbb{Z})$ is isomorphic to $i(F) * i(\mathbb{Z})$. Thus i is an isomorphism.

To complete the proof of the lemma, we will show that for a generic choice of t , the composite of i with the induced map $\mathrm{SL}_2 K \rightarrow \mathrm{SL}_2 \mathbb{C}$ is also injective. Let $w \in F * \mathbb{Z}$ and consider the set of t for which w maps to the identity under the map $F * \mathbb{Z} \rightarrow G_t$. Such t are the solutions to some polynomial equations in t ; as i is injective, these equations are non-trivial and have a finite solution set. As F is finitely generated, there are only countably many such $w \in F * \mathbb{Z}$ and hence only countably many “bad” values for t . Thus if we select $t \in \mathbb{C}$ from outside this countable set we have that G_t is the free product $F * \mathbb{Z}$, as desired. \square

2.1. Remark. If, as is the case in our application, the entries of the elements of F live in some number field L , then one can simply take t to be any transcendental number, e.g. π or e .

3. EXAMPLE

Our example of a fibered 3-manifold satisfying Requirement 1.4 is the orientable double cover of the hyperbolic manifold $M = y505$ from the Callahan-Hildebrand-Weeks census [CHW] that comes with SnapPea [W]. The manifold M is non-orientable and has one torus cusp. It has an ideal triangulation with 7 tetrahedra. The shapes of these tetrahedra all lie in the field $\mathbb{Q}(e^{\pi i/3})$. The fundamental group of M has a presentation

$$\pi_1(M) = \langle a, b \mid aabbaaBAbaBabAABBAbAAB = 1 \rangle,$$

where $A = a^{-1}$ and $B = b^{-1}$. The *faithful* action of $\pi_1(M)$ on \mathbb{CP}^1 is given by

$$a: z \mapsto 1 + \frac{1 + \omega^2}{\bar{z}} \text{ and } b: z \mapsto \frac{-3\bar{\omega}z + 9\omega - 6\bar{\omega}}{(\omega + \bar{\omega})z - 7\omega + 5\bar{\omega}},$$

where $\omega = e^{\pi i/6}$, a primitive 12th root of unity. Note here that a is orientation reversing, and that expressed as a matrix

$$a^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 + \omega^2 & 2 - \omega^2 \\ 1 & 2 - \omega^2 \end{pmatrix}$$

which has trace $\sqrt{3} + 1/\sqrt{3}$, as desired.

In a minute, we will pass to the orientation cover N of M to produce the needed example, but first let us observe that M fibers over the circle. First note that the defining relation for $\pi_1(M)$ is in the commutator subgroup of the free group on a

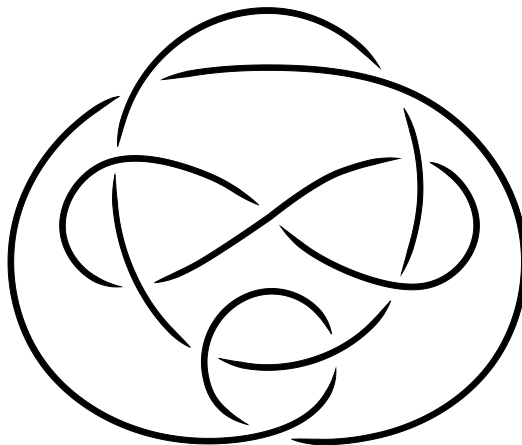


FIGURE 3.1. Doing 0 surgery on the circular component with no self-crossings yields the manifold N . This is link 10_5^3 in the Christy table of links through 10 crossings.

and b , and hence $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Consider the homomorphism $\phi: \pi_1(M) \rightarrow \mathbb{Z}$ given by $\phi(a) = 1$ and $\phi(b) = 0$. If we set $b_k = a^k b a^{-k}$ and $B_k = b_k^{-1}$ we see that the defining relation of $\pi_1(M)$ becomes

$$b_2^2 B_4 b_3 B_4 b_5 B_3^2 b_2 B_0 = 1.$$

Since the highest and lowest subscripts appear only once each, we see by Magnus rewriting [MKS] that the kernel of ϕ is freely generated by $\{b_0, b_1, b_2, b_3, b_4\}$. Thus by a theorem of Stallings [Sta], the map $M \rightarrow S^1$ induced by ϕ can be homotoped to a fibration where the fundamental group of the fiber is the kernel of ϕ . (Actually, for Theorem 1.1, we do not really need Stallings' theorem since we have explicitly exhibited the desired free normal subgroup. However, ultimately the justification that this subgroup of matrices is free comes from identifying the full group of matrices with $\pi_1(M)$; that is, the representation given above is faithful. This is seen by deriving the representation from a hyperbolic structure on M , described by a geometric solution to the gluing equations for some ideal triangulation.)

Now pass to the orientable double cover N of M , whose fundamental group is $\phi^{-1}(2\mathbb{Z})$. (For a Dehn surgery description of N , see Figure 3.1.) Note that the kernel of ϕ restricted to $\pi_1(N)$ has not changed and is still the free group F of rank 5. Moreover, a^2 is not in F as $\phi(a^2) = 2$ and, as we mentioned, the trace of a^2 is $\sqrt{3} + 1/\sqrt{3}$. Thus N satisfies Requirement 1.4, and this completes the proof of Theorem 1.1.

3.2. Remark. It is easy to use Brown's elegant algorithm [Bro] for computing the Bieri-Neumann-Strebel invariant of a 1-relator group to see that any $\phi: \pi_1(M) \rightarrow \mathbb{Z}$ except b^* and $a^* - 2b^*$ corresponds to a fibration. (Here $\{a^*, b^*\}$ is the dual basis of $H^1(M; \mathbb{Z})$ with respect to the basis $\{a, b\}$ of $H_1(M; \mathbb{Z})$.) This gives infinitely many other examples, where the rank of the free group in question grows more slowly than in examples obtained by taking subgroups of finite index in the kernel of a^* .

3.3. Remark. Of the 6070 hyperbolic 3-manifolds in the Callahan-Hildebrand-Weeks census, $y505$ seems to be the only one that satisfies Requirement 1.4. There are a

handful of other examples (mostly also non-orientable) which have an element with the right trace, but these are not fibered. However, it is plausible that they have finite covers which are fibered.

ACKNOWLEDGMENTS

Both authors were partially supported by the U.S. National Science Foundation (grant #DMS-0405491) and the Sloan Foundation.

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